Robust post-accident reconstruction of loading forces

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Abstract. The paper presents a novel methodology for robust post-accident reconstruction of spatial and temporal characteristics of the load. The methodology is based on analysis of local structural response, and identifies an observationally equivalent load, which in a given sense optimally approximates the real load. Compared to previous researches this approach allows to use a limited number of sensors to reconstruct general dynamic loads of unknown locations, including multiple impacts and moving loads. Additionally, the problem of optimum sensor location is studied.

Introduction

The motivation for this research is the need for a general analysis technique for efficient reconstruction of the scenario of a sudden load (e.g. impact, collision etc.), which could be used in a black box type systems. The methodology is based on analysis of local structural response and is applicable to all impact-exposed engineering structures, provided a dedicated sensors system is distributed in the structure in order to measure and store local response.

The technique identifies the modally equivalent load, which is observationally indistinguishable from the real load and optimum in a given sense. The reconstruction is in fact a deconvolution problem and can be formulated analytically as a complex optimisation problem: find the equivalent impact scenario that (i) minimises the mean-square distance between simulated and measured dynamic responses in sensor locations and (ii) is optimum in a given sense. The reconstruction quality is directly related to the number and location of sensors, hence a part of this paper proposes two complementary criteria of correct sensor location.

Compared to previous researches [1,2] this approach allows to reconstruct with a highly limited number of sensors general dynamic loads of unknown locations, including simultaneous multiple impacts and moving loads.

Additionally, numerical examples illustrating the methodology are presented.

Problem Formulation

Linear Response to Dynamic Loading. Let the system being considered be linear and spatially discretised. Provided both the excitation and the system transfer function are known, response of the system in a given sensor location can be expressed by means of a convolution integral as follows:

$$\varepsilon_\alpha(t) = \sum_n \int_{-\varDelta T}^{t} B_{\alpha n}(t-\tau) p_n(\tau) d\tau, \tag{1}$$

where ε_α and p_n represent respectively linear system response in the α-th location and loading in the n-th degree of freedom (DOF), $B_{\alpha n}$ is the system transfer function relating the response in the α-th location to a Dirac-type loading in a the n-th DOF, and $\varDelta T$ is the maximum system response time (i.e. the maximum time of elastic wave propagation between a loading-exposed DOF and a sensor). Notice that due to the intended limited number of sensors, the considered system is rarely collocated, hence $\varDelta T > 0$. A zero excitation is assumed prior to the time $-\varDelta T$.

In real-world applications continuous responses are rarely known, hence Eq. (1) should be discretised with respect to time to take the form of a linear equation:

$$\varepsilon_{\alpha} = \left\langle \sum_n \sum_{\tau=-\Delta T}^{t} B_{\alpha n}(t-\tau)\, p_n(\tau)\, d\,\tau \,\middle|\, t=0,\Delta t,2\,\Delta t,\dots,T\,\Delta t \right\rangle = \sum_n \boldsymbol{B}_{\alpha n}\, \boldsymbol{p}_n, \tag{2}$$

where ε_α and p_n are respectively vector of responses in the α-th sensor location and vector of loading forces in the n-th DOF in all discretised time instances, T is the number of measurement time steps, and $\boldsymbol{B}_{\alpha n}$ is the Teoplitz matrix corresponding to the α-th sensor location and the n-th DOF.

Load Reconstruction. The idea behind the proposed load reconstruction scheme is basically the one of a deconvolution: compare the measured (ε^M) and the modelled (ε) system responses and obtain the excitation by solving the resulting equation. In the continuous time case it leads to a system of A Fredholm integral equations of the first kind,

$$\varepsilon_\alpha^M(t) = \sum_{n=1}^{N} \int_{-\Delta T}^{t} B_{\alpha n}(t-\tau)\, p_n(\tau)\, d\,\tau, \quad \alpha=1,2,\dots,A, \tag{3}$$

where N is the number of potentially load-exposed DOFs, and in the discrete time case to

$$\varepsilon^M = \boldsymbol{B}\, \boldsymbol{p}, \tag{4}$$

a large linear system, where the vectors ε^M and $\boldsymbol{p}$ contain all the measured discrete responses and all the discretised loading forces, and the block matrix $\boldsymbol{B}$ is composed of all Toeplitz matrices $\boldsymbol{B}_{\alpha n}$.

Notice that in the intended practical situation $A<N$ (fewer sensors than potentially loading-exposed DOFs) both systems Eq. (3) and Eq. (4) are underdetermined. In the case of the discretised system Eq. (4) the reason is twofold: (i) $A<N$ and (ii) time intervals of different length (measurement T and reconstruction $T+\Delta T$) being discretised with the same time step Δt.

Notice also that Eq. (3) and Eq. (4) tend to be ill-conditioned, which is mainly due to the inherent ill-conditioning of Fredholm integral equations of the first kind with a compact kernel, which cannot have a bounded inverse [3]. A seemingly contradictory behaviour is the result: the finer the time discretisation Δt, the more ill-conditioned Eq. (4) is. Moreover, ill-conditioning often arises also due to small (or neglected) time shift ΔT, which results in almost triangular matrices $\boldsymbol{B}_{\alpha n}$ with very small values on the diagonal. Therefore, a regularisation technique is usually a must [2,3,4].

Overdetermined systems

Objective function. If the system Eq. (4) is overdetermined, a unique load $\boldsymbol{p}$ can be found to minimise the following objective function

$$f_M(\boldsymbol{p}) = \left\| \boldsymbol{M}\left(\varepsilon^M - \boldsymbol{B}\,\boldsymbol{p}\right) \right\|^2, \tag{5}$$

where $\boldsymbol{M}$ is a preconditioner matrix. For reasons of simplicity a unity preconditioner matrix $\boldsymbol{M}=\boldsymbol{I}$ will be assumed further on. Taking into account eventual regularisation, the objective function can be rewritten as

$$f_I(\boldsymbol{p}) = \sum_{t=0}^{T} \sum_{\alpha=1}^{A} \left[\varepsilon_\alpha^M(t) - \varepsilon_\alpha(t) \right]^2 + \delta \| \boldsymbol{D}\,\boldsymbol{p} \|^2, \tag{6}$$

where $\delta >= 0$ is an eventual Tikhonov regularisation term [2,3,4]. Due to Eq. (2) the derivative of the objective function f_I with respect to $p_n(t)$ is expressible as

$$\frac{\partial f_I(\boldsymbol{p})}{\partial p_n(t)} = -2 \sum_{\tau=t}^{T} \sum_{\alpha=1}^{A} \left[\varepsilon_\alpha^M(\tau) - \varepsilon_\alpha(\tau) \right] B_{\alpha n}(\tau-t), \tag{7}$$

where $\delta = 0$ is assumed for notational simplicity.

Basic formulae. The objective function is easily verified to be quadratic and convex, hence it can be expanded around a given loading vector $\boldsymbol{p}$ as

$$f_I(\boldsymbol{p}+\boldsymbol{d})=f_I(\boldsymbol{p})+\nabla f_I(\boldsymbol{p})^T\boldsymbol{d}+\frac{1}{2}\boldsymbol{d}^T\boldsymbol{H}\boldsymbol{d}. \tag{8}$$

Eq. (8) compared with with Eq. (6) and Eq. (7) leads to the two following basic formulae:

$$\nabla f(\boldsymbol{p})^T\boldsymbol{d}=-2\sum_{t=0}^{T}\sum_{\alpha\in\Sigma}\varepsilon_\alpha^{(d)}(t)\left[\varepsilon_\alpha^M(t)-\varepsilon_\alpha^{(p)}(t)\right],$$
$$\boldsymbol{d}_i^T\boldsymbol{H}\boldsymbol{d}_j=2\sum_{t=0}^{T}\sum_{\alpha\in\Sigma}\varepsilon_\alpha^{(d_i)}(t)\cdot\varepsilon_\alpha^{(d_j)}(t). \tag{9}$$

Line optimisation. To avoid computing and inverting the Hessian of the objective function, which would be much more ill-conditioned than $\boldsymbol{B}$, optimisation has to consist of a series of line optimisations. Each one amounts to finding the minimum at a given loading $\boldsymbol{p}$ along the direction $\boldsymbol{d}$, i.e. the minimum of $f_i(\boldsymbol{p}+s\boldsymbol{d})$ with respect do s, which due to Eq. (8) is a convex quadratic function with the (easily calculable by Eq. (9)) minimum at

$$s_{min}=-\frac{\nabla f_I(\boldsymbol{p})^T\boldsymbol{d}}{\boldsymbol{d}^T\boldsymbol{H}\boldsymbol{d}}. \tag{10}$$

Conjugate gradient. Eq. (7) allows to find the steepest descent direction $-grad\,f_I$. However, the steepest descent method suffers from slow convergence. The objective function is unbounded quadratic, thus perfectly suited for the conjugate gradient method: choosing in each optimisation step a direction $\boldsymbol{d}_{n+1}$ *conjugate* to all previous directions $\boldsymbol{d}_0, ..., \boldsymbol{d}_n$ leads by Eq. (10) directly to the minimum in the subspace generated by all considered directions. Therefore, starting with the steepest descent direction and making use of the conjugacy criterion $\boldsymbol{d}_i^T\boldsymbol{H}\boldsymbol{d}_j = 0$,

$$\boldsymbol{d}_{n+1}=-\nabla f(\boldsymbol{p}_{n+1})+\sum_{i=0}^{n}\eta_{n+1,i}\boldsymbol{d}_i,\text{where }\eta_{n+1,i}=\frac{\nabla f(\boldsymbol{p}_{n+1})^T\boldsymbol{H}\boldsymbol{d}_i}{\boldsymbol{d}_i^T\boldsymbol{H}\boldsymbol{d}_i}. \tag{11}$$

Theoretically, there should stand in Eq. (11) $\eta_{n+1,i} = 0$ for $i < n$ [5]. However, the limited accuracy of the floating point arithmetic leads to correction terms with respect to all previous directions, Eq. (11) is thus a counterpart of the Gram-Schmidt orthogonalisation scheme.

Underdetermined systems

All known research [1,2] deals with the overdetermined case only, hence heavily limits the generality of the load being reconstructed, which is usually assumed to be a non-moving, single force pinpointing a single DOF. The location of the affected DOF is known in advance or determined in a second-stage non-linear optimisation. However, the approach of this paper addresses directly the general underdetermined case.

Load decomposition. Matrix $\boldsymbol{B}$ of Eq. (4) has a unique singular value decomposition (SVD) [6],

$$\boldsymbol{B}=\boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^T, \tag{12}$$

where $\boldsymbol{U}$ and $\boldsymbol{V}$ are (square) unitary matrices ($\boldsymbol{U}^T\boldsymbol{U}=\boldsymbol{I}$ and $\boldsymbol{V}^T\boldsymbol{V}=\boldsymbol{I}$) and $\boldsymbol{\Sigma}$ is a diagonal matrix. Let $\boldsymbol{P}$ be the linear space of all possible load configurations, and let $\boldsymbol{V}_1$ and $\boldsymbol{V}_2$ denote the two component matrices of $\boldsymbol{V}=[\boldsymbol{V}_1\ \boldsymbol{V}_2]$, where the number of columns of $\boldsymbol{V}_1$ equals the number of positive singular values of $\boldsymbol{B}$. Notice that the columns of $\boldsymbol{V}_1$ and $\boldsymbol{V}_2$ are orthonormal vectors in $\boldsymbol{P}$ and span two orthogonal and complimentary linear subspaces $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$, $\boldsymbol{P}=\boldsymbol{P}_1\ x\ \boldsymbol{P}_2$. Due to Eq. (12) $\boldsymbol{B}\boldsymbol{V}_2=\boldsymbol{0}$, i.e. the linear operator $\boldsymbol{B}$ projects $\boldsymbol{P}$ onto $\boldsymbol{P}_1$ to transform it later orthonormally via $\boldsymbol{U}$. Therefore, $\boldsymbol{P}_1$ is the reconstructible subspace of $\boldsymbol{P}$ with respect to $\boldsymbol{B}$, while $\boldsymbol{P}_2$ is the unreconstructible subspace of $\boldsymbol{P}$ (with respect to $\boldsymbol{B}$). In other words, each load can be decomposed

into a sum of two orthogonal components, the first p_R being a linear combination m_1 of columns of V_1 and fully reconstructible from the (exact) measurements, which retain no information concerning the second component being a linear combination of columns of V_2:

$$p = V_1 m_1 + V_2 m_2 = p_R + V_2 m_2,$$
$$B\, p = B\, p_R = U\, \Sigma\, m_1. \qquad (13)$$

Reconstructible load component. Given the measurements ε^M, the corresponding reconstructible load component p_R can be found either using the standard pseudoinverse of B,

$$p_R = V\, \Sigma^+ U^T \varepsilon^M, \qquad (14)$$

where Σ^+ is the transposed diagonal matrix containing reciprocals of the corresponding positive singular values and zeros, or by the optimisation technique described above for the overdetermined systems. The latter approach is quicker, as it does not require performing the SVD, but it renders estimation of the irreconstructible load component impossible (since it requires knowledge of V_2). Nevertheless, the quickness argument becomes less important in the case of multiple load reconstructions: the SVD has to be calculated only once.

Notice that due to strong ill-conditioning of B, some of the diagonal elements Σ^+ can be very large and may have to be reduced or zeroed, which is a way to regularise the solution [2,4].

Unreconstructible load component. According to Eq. (13) any linear combination of columns of V_2 added to the reconstructible load component does not change the response of the sensors. Hence the loadings of the form

$$p = p_R + V_2 m_2, \qquad (15)$$

where m_2 is a vector of arbitrary coefficients, are all admissible solutions to Eq. (4). Further choice of the *optimum* loading must be therefore based on additional criteria, not related to the measurements ε^M. It can be based on an *a priori* knowledge of expected characteristics od the loading: from all admissible loadings Eq. (15) choose the one that minimises a given norm D (e.g. the derivative D_1 with respect to time and/or space to obtain smooth loadings), i.e. minimise

$$g_D(m_2) = \left\| D \left(p_R + V_2 m_2 \right) \right\|^2. \qquad (16)$$

Notice that (i) if $D = I$, then the g_D is minimised by $m_2 = 0$ and p_R is the optimum, and (ii) g_D is a quadratic function of m_2, hence the optimum loading p_D depends linearly on p_R and ε^M,

$$p_D = \left[I - V_2 \left(V_2^T D^T D V_2 \right)^{-1} V_2^T D^T D \right] p_R = X_D \varepsilon^M. \qquad (17)$$

Optimal Sensor Location

Optimality criteria. If the number of potentially load-exposed DOFs N exceeds considerably the number of available sensors A, which is intended in this paper, the question of optimum sensor location arises. There is not much theoretical investigation into the problem. Mackiewicz at al. [7] propose to locate the sensors to minimise the ill-conditioning of the reconstruction process. This can be here quantified as the task of finding the sensor location π, which minimises the following standard measure of ill-conditioning of the corresponding matrix B_π:

$$q_1(\pi) = -\log \left[\sigma_{max}(B_\pi) \,/\, \sigma_{min}(B_\pi) \right]. \qquad (18)$$

In Eq. (18) π is a location of all A sensors thorough the structure, hence it can be represented as an

A-element subset of $\{1, 2, ..., A_{max}\}$, where A_{max} is the number of all possible locations of a single sensor. σ_{max} and σ_{min} are the maximum and minimum singular values of the corresponding matrix $\mathbf{B}_\pi$.

In underdetermined systems considered here arises also the problem of the reconstruction accuracy, which for a given loading can be quantified as the distance between the loading and its reconstructible component. Thus, if $V_{1\pi}$ and $V_{2\pi}$ denote the matrices V_1 and V_2 calculated for a given sensor location π, then a measure of reconstruction accuracy with respect to a given set of expected loadings $\{\boldsymbol{p}_1, \boldsymbol{p}_2, ..., \boldsymbol{p}_M\}$ can be defined as

$$q_2(\pi)=\sum\nolimits_{i=1}^{M} \left\| \left(\boldsymbol{I}-\boldsymbol{V}_{1\pi}\boldsymbol{V}_{1\pi}^T\right)\boldsymbol{p}_i \right\|^2 \;=\; \sum\nolimits_{i=1}^{M}\left\|\boldsymbol{V}_{2\pi}\boldsymbol{V}_{2\pi}^T\boldsymbol{p}_i\right\|^2. \tag{19}$$

Numerical example.

Fig. 1 shows the 10 element, 1 m long frame structure used in the numerical example. Loading forces can occur only vertically in any/all of the 10 nodes. The measurement time interval is $T = 10$ ms, discretised into 100 time steps, the time shift is $\Delta T = 0.5$ ms, hence the reconstruction time interval is 10.5 ms long (105 time steps). Three or four sensors can be located in any of the 10 elements (A equals 3 or 4), each measuring the local curvature (the difference between the two neighbouring rotational DOFs). The matrix $\mathbf{B}$ is thus 300 x 1050 or 400 x 1050.
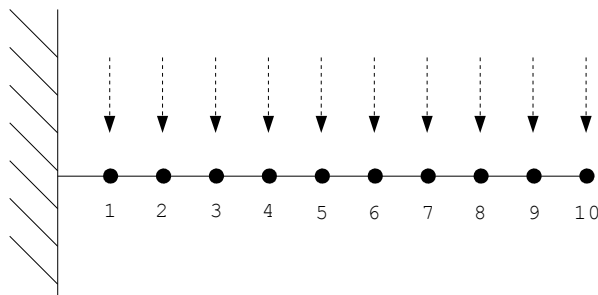

Fig. 1 Frame structure used in the numerical example (steel, 200 GPa, 7800 kg/m^3, $\varphi = 1$ cm, $l = 1$ m)
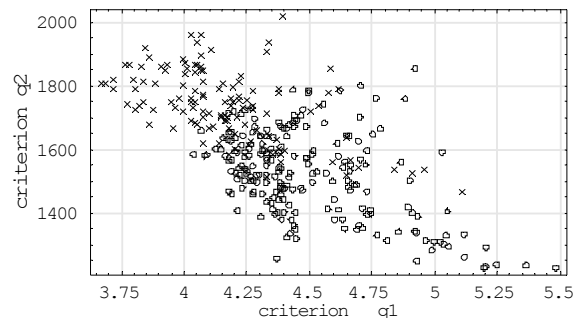

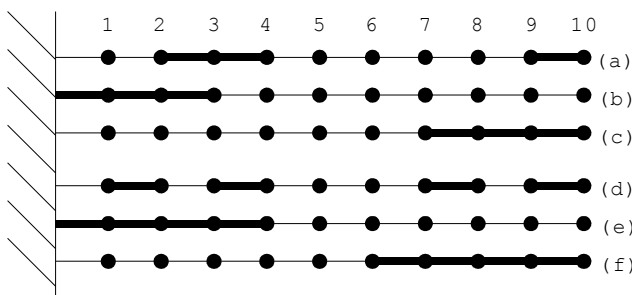**Fig. 2** Correlation plot for sensor location criteria (crosses – 3 sensors, circles – 4 sensors)

Fig. 3 Best and worst sensor locations:
3 sensors: (a) q_1 best; (b) q_1 worst, q_2 best; (c) q_2 worst
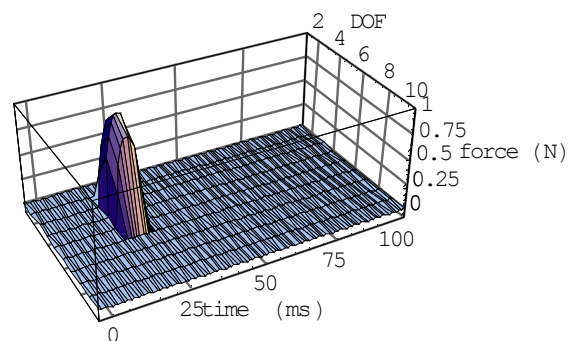4 sensors: (d) q_1 best; (e) q_1 worst, q_2 best; (f) q_2 worst

Fig. 4 One of 168 identical in shape loadings used for calculating criterion q_2

The diagram in Fig. 2 illustrates the correlation between the sensor location criteria q_1 and q_2, each dot corresponds to one sensor location; Fig. 3 shows the calculated best and worst locations. The criterion q_2 has been calculated with respect to a set of 168 loadings of the same shape (a sample is depicted in Fig. 4), the half being distributed uniformly and the rest randomly in time-space. Fig. 2 shows strong negative correlation between q_1 and q_2 ($\rho = -0.66$), which is also confirmed in Fig. 3: worst sensor locations with respect to q_1 are best wrt q_2. The q_2-best locations may seem astonishing unless realised that they are the locations with the largest responses (local curvature).

The reconstructible components of the test loading shown in Fig. 4 are calculated wrt the q_2-best and q_1-best locations of four sensors and shown respectively in Fig. 5 and Fig. 6. The former is far more accurate, but also much more prone to measurement errors. A real-world criterion for sensor location should thus take into account reconstruction accuracy, but with the number of singular values (i.e. columns of V_1) limited according to the expected level of measurement error.
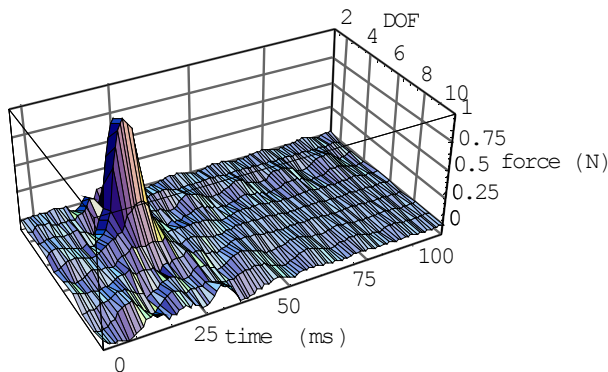

Fig. 5 Reconstructible load component, four sensors, q_2-best sensor location

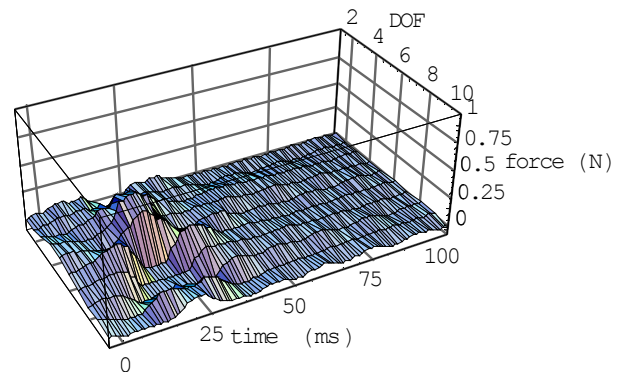**Fig. 6** Reconstructible load component, four sensors, q_1-best sensor location

Conclusions and Further Steps

A robust methodology for a posteriori reconstruction of loading forces is described. Its novelty lies in the limitation of the number of sensors necessary to reconstruct a general dynamic loading, including multiple and moving load cases. Potential application area are black box type systems.

The research is ongoing to investigate further the problem of optimum sensor location and to include the case of small plastic deformations. Experimental verification is also being prepared.

Acknowledgments

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